Nonhomogenous, Linear, Second–Order, Differential Equations

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Seminar in Engineering Analysis

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California State University Northridge

Outline

- Review last class
- Second-order nonhomogenous equations with constant coefficients
  - Solution is sum of homogenous equation solution, y_H , plus a particular solution, y_P, for the nonhomogenous part
  - Method of undetermined coefficients
  - Variation of parameters

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Review $y'' + \alpha y' + \beta y = 0$

- Three cases depending on $\omega^2 = \beta - \alpha^2/4$
- Double root when $\beta = \alpha^2/4$:
  - $y = (C_1 + C_2 x) e^{-\alpha x/2}$
- Complex roots when $\beta > \alpha^2/4$, $\omega^2 > 0$
  - $y = e^{-\alpha x/2} [A\cos \omega x + B\sin \omega x]$
- Distinct real roots when $\beta < \alpha^2/4$
  - $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

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Review $y'' + \alpha y' + \beta y = 0$ II

- Initial conditions $y(0) = y_0$ and $y'(0) = v_0$
- Double root when $\beta = \alpha^2/4$:
  - $y = [(v_0 + y_0\alpha/2)x + y_0] e^{-\alpha x/2}$
- Complex roots when $\beta > \alpha^2/4$
  - $y = e^{-\alpha x/2} [y_0 \cos \omega x + \omega^{-1}(v_0 + y_0\alpha/2)\sin \omega x]$
- Distinct real roots when $\beta < \alpha^2/4$
  - $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
  - $C_1 = (\lambda_2 y_0 - v_0)/(\lambda_2 - \lambda_1)$
  - $C_2 = (v_0 - \lambda_1 y_0)/(\lambda_2 - \lambda_1)$

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Example: Mechanical Systems

- Spring/mass/damper with $md^2y/dt^2 + cdy/dt + ky = 0$, $y(0) = y_0$, $y'(0) = v_0$
- Solutions depended on parameters
  - Pure oscillation for c = 0
  - Underdamping when $c^2 < 4km$
  - Critical damping when $c^2 = 4km$
  - Overdamping when $c^2 > 4km$
  - Solution for y/y_0 depends on ct/m (or ωt for c = 0; $\omega^2 = k/m$), km/c^2, and mv_0/cy_0
  - c^2/m^2 and k/m have dimensions $(\text{time})^{-2}$

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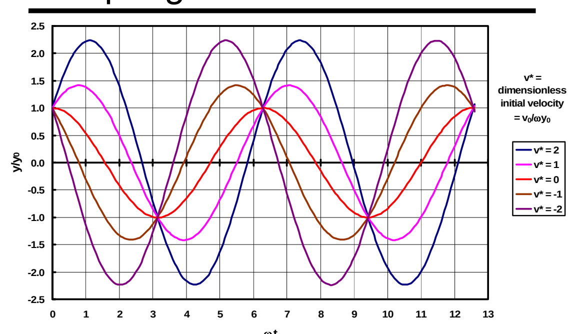
Spring-mass Oscillations



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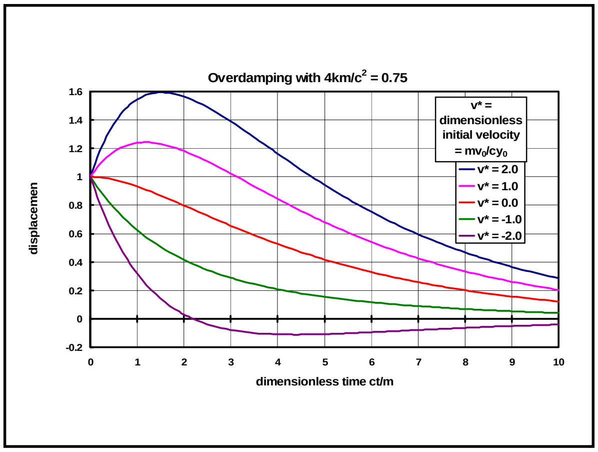

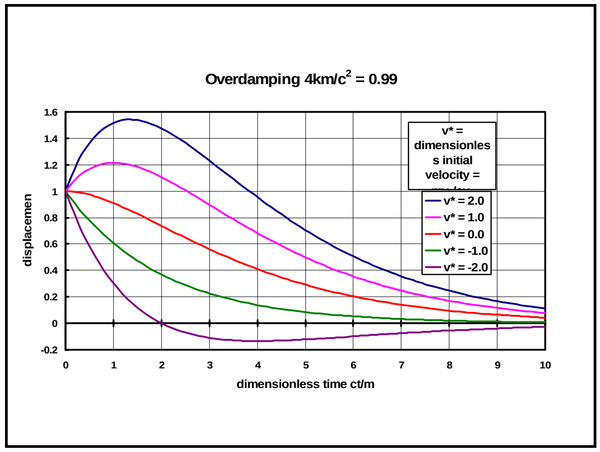

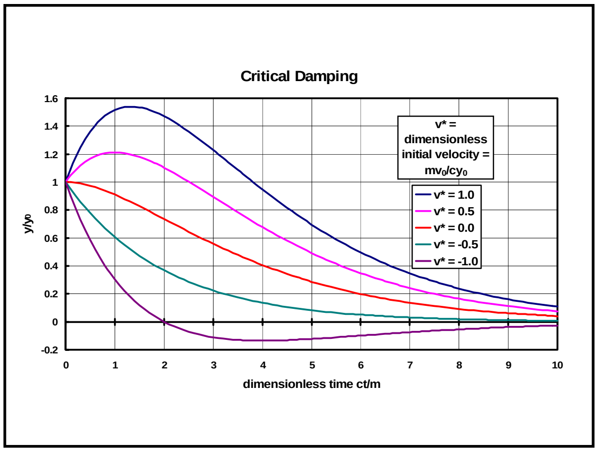

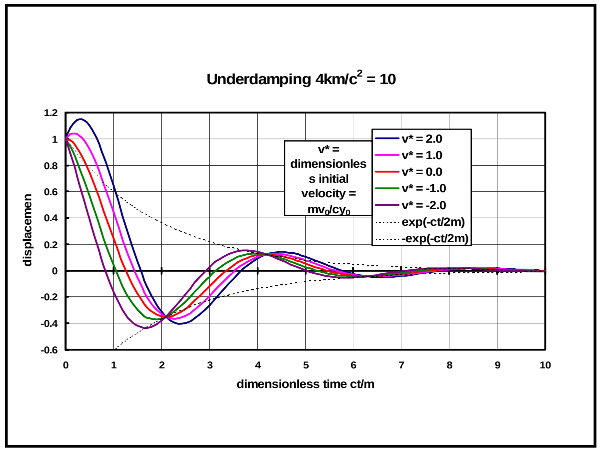

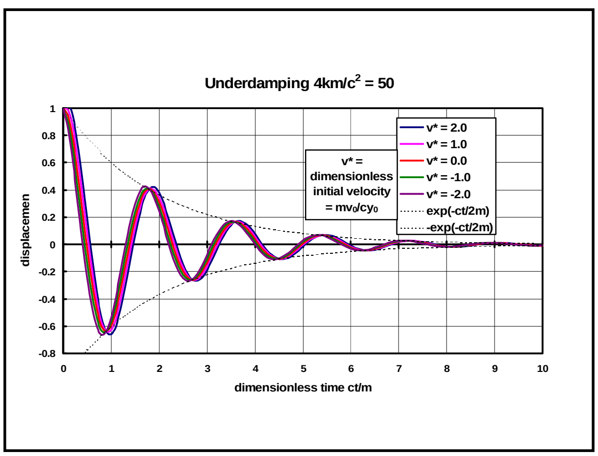

Alternative Formulations

- Proposed alternative with equation from trigonometry for cos(a – b)

$$y = C\cos(\omega t - \delta) = C\sin\delta\sin\omega t + C\cos\delta\cos\omega t$$

- How does this match original equation?

$$y = A\sin\omega t + B\cos\omega t$$

- Matches if A = C sin δ and B = C cos δ
- This gives $A^2 + B^2 = C^2 \sin^2\delta + C^2 \cos^2\delta = C^2$ and $A/B = (C \sin\delta)/(C \cos\delta) = \tan\delta$ or $\delta = \tan^{-1}(A/B) = \tan^{-1}(v_0/y_0\,\omega)$

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Nonhomogeneous Equations

- Solution to linear nonhomogeneous second-order equation, $y = y_H + y_P$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

- y_H is general solution to corresponding homogenous equation, $C_1y_1 + C_2y_2$

$$\frac{d^2y_H}{dx^2} + p(x)\frac{dy_H}{dx} + q(x)y_H = 0$$

- y_P is particular solution

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Nonhomogeneous Equations II

- What is first step in solving the general second-order nonhomogeneous ODE?

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

- First step is finding y_H, which is the general solution to corresponding homogenous equation

$$\frac{d^2y_H}{dx^2} + p(x)\frac{dy_H}{dx} + q(x)y_H = 0$$

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Nonhomogeneous Equations III

- Substitute solution $y = y_H + y_P$ into the nonhomogeneous equation

$$\frac{d^2(y_H + y_P)}{dx^2} + p(x)\frac{d(y_H + y_P)}{dx} + q(x)(y_H + y_P) = r(x)$$

$$\frac{d^2y_H}{dx^2} + \frac{d^2y_P}{dx^2} + p(x)\frac{dy_H}{dx} + p(x)\frac{dy_P}{dx} + q(x)y_H + q(x)y_P = r(x)$$

$$\boxed{\frac{d^2y_H}{dx^2} + p(x)\frac{dy_H}{dx} + q(x)y_H} + \frac{d^2y_P}{dx^2} + p(x)\frac{dy_P}{dx} + q(x)y_P = r(x)$$

Homogeneous Equation = 0

$$\frac{d^2y_P}{dx^2} + p(x)\frac{dy_P}{dx} + q(x)y_P = r(x)$$

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Nonhomogeneous Equations IV

- Solve nonhomogeneous equations by first finding homogeneous solution

$$\frac{d^2y_H}{dx^2} + p(x)\frac{dy_H}{dx} + q(x)y_H = 0$$

- Do **not** solve for C_1 and C_2 in homogeneous solution $y_H = C_1y_1 + C_2y_2$
- Solve $\frac{d^2y_P}{dx^2} + p(x)\frac{dy_P}{dx} + q(x)y_P = r(x)$ for y_P
- Apply initial/boundary conditions to $y = C_1y_1 + C_2y_2 + y_P$ to get C_1 and C_2

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Solving $\frac{d^2y_P}{dx^2} + \alpha\frac{dy_P}{dx} + \beta y_P = r(x)$

- Method of undetermined coefficients applies for constant coefficient equation
  - Assume a solution for y_P based on the form of r(x) with constants
    - Process for assuming y_P to be described later
  - E.g., if $r(x) = x^2$ assume a solution of the form $y_P = a_0 + a_1 x + a_2 x^2$
  - Substitute proposed solution into the differential equation for y_P
  - Set coefficient sum of like terms in y_P to zero
  - Solve for the unknown a_k coefficients in y_P

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Example: solve $\frac{d^2y_P}{dx^2} + 3\frac{dy_P}{dx} + 2y_P = x^2$

- Assume $y_P = a_0 + a_1 x + a_2 x^2$
- Substitute y_P, $dy_P/dx = a_1 + 2a_2x$ and $d^2y_P/dx^2 = 2a_2$ into differential equation

$$2a_2 + 3(2a_2x + a_1) + 2(a_2x^2 + a_1x + a_0) = x^2$$

- Set coefficient sums of like terms (like powers of x in this example) to zero
- $2a_2 + 3a_1 + 2a_0 = 0$ for x^0 terms
- $6a_2 + 2a_1 = 0$ for x^1 terms
- $2a_2 = 1$ for x^2 terms

3 equations for a_0, a_1, and a_2 solved on next slide

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Example: solve $\frac{d^2y_P}{dx^2}+3\frac{dy_P}{dx}+2y_P=x^2$

- $2a_2 + 3a_1 + 2a_0 = 0 \Rightarrow a_0 = -a_2 - 3a_1/2$
- $6a_2 + 2a_1 = 0 \Rightarrow a_1 = -3a_2$
- $2a_2 = 1 \Rightarrow a_2 = ½$ so $a_1 = -3a_2 = -3/2$
- $a_0 = -a_2 - 3a_1/2 = -1/2 – 3(-3/2)/2 = 7/4$
- So $y_P = a_0 + a_1 x + a_2 x^2 = 7/4 – 3x/2 + x^2/2$
- Check this: $y_P'' + 3y_P' + 2y_P = 1 + 3(x – 3/2) + 2(7/4 – 3x/2 + x^2/2) = 1 – 9/2 + 7/2 + x(3 – 3) + x^2 = x^2$ proving solution for y_P

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Example: solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x^2$

- Just found $y_P = 7/4 – 3x/2 + x^2/2$
- Characteristic equation for homogenous ODE has two distinct real roots

$$\lambda=\frac{-\alpha\pm\sqrt{\alpha^2-4\beta}}{2}=\frac{-3\pm\sqrt{3^2-4(2)}}{2}=-2,-1$$

- $y = y_H + y_P = C_1e^{-2x} + C_2e^{-x} + 7/4 – 3x/2 + x^2/2$
- Find C_1 and C_2 from initial conditions such as $y(0) = y_0$ and $y'(0) = v_0$

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Example: solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x^2$

- $y = C_1e^{-2x} + C_2e^{-x} + 7/4 – 3x/2 + x^2/2$
- $y' = -2C_1e^{-2x} - C_2e^{-x} – 3/2 + x$
- $y_0 = C_1e^0 + C_2e^0 + 7/4 – 3(0)/2 + 0^2/2$
- $y_0 = C_1 + C_2 + 7/4 \Rightarrow C_2 = y_0 – 7/4 – C_1$
- $v_0 = –2C_1e^0 – C_2e^0 – 3/2 + 0$
- $v_0 = –2C_1 – C_2 – 3/2$
- $y_0 + v_0 = C_1 + C_2 + 7/4 + (– 2C_1 – C_2 – 3/2)$
- $C_1 = ¼ – y_0 – v_0$
- $C_2 = y_0 – 7/4 – C_1 = 2y_0 + v_0 – 2$

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Example: solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=x^2$

- Summary of process to solve nonhomogeneous ODE with given initial conditions
  - First found $y_H = C_1e^{-2x} + C_2e^{-x}$
  - Next found $y_P = 7/4 – 3x/2 + x^2/2$
  - Wrote $y = y_H + y_P$ (including C_1 and C_2)
  - Then used initial conditions on y and dy/dx to find $C_1 = ¼ – y_0 – v_0$ and $C_2 = 2y_0 + v_0 – 2$
  - Substitute C_1 and C_2 equations into solution for y to get desired result
  - $y = (¼ – y_0 – v_0)e^{-2x} + (2y_0 + v_0 – 2)e^{-x} + 7/4 – 3x/2 + x^2/2$

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Finding Particular Solution

- Used for constant coefficient equation $y'' + ay' + by = r(x)$
- Solution is $y = y_P + y_H$, where y_H is solution of $y_H'' + ay_H' + by_H = 0$
- Postulate a solution for y_P following gidelines on next two charts
- Plug postulated y_P solution into ODE and solve for unknown coefficients
  – Overall coefficients of like terms on both sides of ODE must vanish

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Table of Trial y_P Solutions

For these r(x)	Start with this y_P
$r(x) = Ae^{ax}$	$y_P = Be^{ax}$
$r(x) = Ax^n$	$y_P = a_0 + a_1x + … + a_nx^n$
$r(x) = A\sin \omega x$	$y_P = B \sin \omega x + C \cos \omega x$
$r(x) = A\cos \omega x$	
$r(x) = Ae^{ax}\sin \omega x$	$y_P = e^{ax} (B \sin \omega x + C \cos \omega x)$
$r(x) = Ae^{ax}\cos \omega x$	

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Special Rules

- If the right-hand-side, r(x) consists of more than one term from the previous table, use a y_P that contains all the corresponding y_P terms
  - For $r(x) = A\cos bx + Ce^{dx}$, use $y_P = E \sin bx + F \cos bx + Ge^{dx}$
- If r(x) is proportional to a solution for the homogenous equation, use y_P equal to x times the y_P shown in the table
  - For a double root, multiply table y_P by x^2

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Exercise: solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2\cos(5x)$

- Previous example with new right hand side
- Keep same initial conditions $y(0) = y_0$ and $dy/dx(0) = v_0$
- Will have same homogenous solution found previously: $y_H = C_1e^{-2x} + C_2e^{-x}$
- Find new particular solution for 2cos(5x)
- Use initial conditions on y and dy/dx to find C_1 and C_2
- Substitute C_1 and C_2 equations into solution for y to get desired result

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Variation of Parameters

- Background – we previously stated the solution to the general first-order linear equation dy/dx + f(x)y = g(x)

$$p = \int f(x)dx \qquad y = e^{-p}\left[C + \int e^{p} g(x)dx\right]$$

- Now we derive this solution as an example of variation of parameters
- Start with solution to homogenous equation: $dy_H/dx + f(x)y_H = 0$

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Variation of Parameters II

- Homogenous equation has separable solution found below

$$\frac{dy_H}{dx} = -f(x)y_H \quad \Rightarrow \quad \frac{dy_H}{y_H} = -f(x)dx$$

$$\ln y_H = -\int f(x)dx \quad \Rightarrow \quad y_H = e^{-p} \qquad p = \int f(x)dx$$

- Try a solution to the nonhomogenous equation of the form $y(x) = u(x)y_H(x)$
- This gives $y' = u'y_H + uy_H'$ dy/dx + f(x)y = g(x)
- Substitute this into original equation

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Variation of Parameters III

- Rearrange equation to get homogenous differential equation with y_H that is zero

$$\frac{dy}{dx} + f(x)y = g(x) \quad \Rightarrow \quad y_H\frac{du}{dx} + u\frac{dy_H}{dx} + f(x)(uy_H) = g(x)$$

$$u\left[\frac{dy_H}{dx} + f(x)y_H\right] + y_H\frac{du}{dx} = g(x) \quad \Rightarrow \quad y_H\frac{du}{dx} = g(x)$$

- Since we know $y_H(x)$, we now have a separable ODE to solve for u as shown on the next chart

$$\ln(y_H) = \int f(x)dx$$

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Variation of Parameters IV

- Rearrange final (separable) differential equation to solve for u by integration

$$y_H\frac{du}{dx} = g(x) \quad \Rightarrow \quad \frac{du}{dx} = \frac{g(x)}{y_H} \quad \Rightarrow \quad u = \int \frac{g(x)}{y_H}dx + C$$

$$\text{Substitute } y = uy_H = ue^{-p} \qquad p = \int f(x)dx$$

$$y = uy_H = y_H\int \frac{g(x)}{y_H}dx + C = e^{-\int f(x)dx}\int e^{\int f(x)dx} g(x)dx + C$$

- Can we apply this to second-order?

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Variation of Parameters V

- Nonhomogenous and homogenous equations $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$
- $y_1(x)$ and $y_2(x)$ are LI homogenous solutions $\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1 = 0$ $\frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2 = 0$
- Try solution $y = u(x)\,y_1(x) + v(x)\,y_2(x)$
  - Two functions but one equation
  - Create another (arbitrary) equation

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Variation of Parameters VI

- First derivative of $y(x)$ is $= u'(x)\,y_1(x) + u(x)\,y_1'(x) + v'(x)\,y_2(x) + v(x)\,y_2'(x)$
- Pick arbitrary second equation to simplify result: $u'(x)\,y_1(x) + v'(x)\,y_2(x) = 0$
- This gives $y'(x) = u(x)\,y_1'(x) + v(x)\,y_2'(x)$
- $y'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$
- Substitute results for y, y', and y'' into the original differential equation

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Variation of Parameters VII

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r \quad\Rightarrow\quad u\frac{d^2y_1}{dx^2} + \frac{du}{dx}\frac{dy_1}{dx} +$$
$$v\frac{d^2y_2}{dx^2} + \frac{dv}{dx}\frac{dy_2}{dx} + p\left[u\frac{dy_1}{dx} + v\frac{dy_2}{dx}\right] + q(uy_1 + vy_2) = r$$

- Rearrange to get homogenous ODE equal zero with solutions y_1 and y_2

$$u\left[\frac{d^2y_1}{dx^2} + p\frac{dy_1}{dx} + qy_1\right] + v\left[\frac{d^2y_2}{dx^2} + p\frac{dy_2}{dx} + qy_2\right]$$ Homogenous ODEs = zero
$$+\frac{du}{dx}\frac{dy_1}{dx} + \frac{dv}{dx}\frac{dy_2}{dx} = r$$

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Variation of Parameters VIII

- Two equations in two unknowns
- ODE Result $\frac{dy_1}{dx}\frac{du}{dx} + \frac{dy_2}{dx}\frac{dv}{dx} = r$
- Auxiliary equation $y_1\frac{du}{dx} + y_2\frac{dv}{dx} = 0$
- Multiply first equation by y_1, second equation by dy_1/dx and subtract

$$\left[y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right]\frac{dv}{dx} = W\frac{dv}{dx} = y_1 r$$

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Variation of Parameters IX

- Repeat last chart with change to get equation for du/dx instead of dv/dx
- ODE Result $\frac{dy_1}{dx}\frac{du}{dx} + \frac{dy_2}{dx}\frac{dv}{dx} = r$
- Auxiliary equation $y_1\frac{du}{dx} + y_2\frac{dv}{dx} = 0$
- Multiply first equation by y_2, second equation by dy_2/dx and subtract first from second

$$\left[y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right]\frac{du}{dx} = W\frac{du}{dx} = -y_2 r$$

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Variation of Parameters X

- Integrate du/dx and dv/dx equations

$$u = -\int\frac{y_2 r(x)}{W(x)}dx \qquad v = \int\frac{y_1 r(x)}{W(x)}dx$$

- Plug into original solution: $y_P = y_1 u + y_2 v$

$$y_P = y_1 u + y_2 v = -y_1\int\frac{y_2 r(x)}{W(x)}dx + y_2\int\frac{y_1 r(x)}{W(x)}dx$$

- Get $y = y_H + y_P$ and evaluate constants in y_H solution from initial conditions

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Nonhomogenous Summary

- Undetermined coefficients is simpler approach but is limited
  - Constant coefficient equations
  - Limited set of functions
- Variation of parameters is more complex, but handles more cases
- In reality, there are no general methods to get homogenous solution to linear, second-order ODE without constant coefficients

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Exercise Solution I

- For right-hand-side 2cos(5x) use particular solution $y_P = A\cos(5x) + B\sin(5x)$
- $dy_p/dx = -5A\sin(5x) + 5B\cos(5x)$
- $d^2y_p/dx = -25A\cos(5x) - 25B\sin(5x)$
- Use ODE for y_P: $y'' + 3y' + 2y = 2\cos(5x)$
- $-25A\cos(5x) - 25B\sin(5x) + 3[-5A\sin(5x) + 5B\cos(5x)] + 2[A\cos(5x) + B\sin(5x)] = 2\cos(5x)$ and set coefficients of sine and cosine terms to zero

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Exercise Solution II

- Equate coefficients of like terms
- $(-25A + 15B + 2A)\cos(5x) = 2\cos(5x)$
- $(-25B - 15A + 2B)\sin(5x) = 0$
- First equation gives $-23A + 15B = 2$
- Second equation gives $A = -23B/15$
- Combined equations $529B/15 + 15B = 2$
- Solving gives $529B + 225B = 30$
- $B = 30/754 = 15/377$; $A = -23B/15 = -23/377$

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Exercise Solution III

- $y_P = A\cos(5x) + B\sin(5x) = -23\cos(5x)/377 + 15\sin(5x)/377$
- $y = y_H + y_P = C_1e^{-2x} + C_2e^{-x} - 23\cos(5x)/377 + 15\sin(5x)/377$
- $y(0) = \boxed{y_0 = C_1 + C_2 - 23/377}$
- $y' = -2C_1e^{-2x} - C_2e^{-x} + 115\sin(5x)/377 + 75\cos(5x)/377$
- $y'(0) = \boxed{v_0 = -2C_1 - C_2 + 75/377}$
- $y_0 + v_0 = -C_1 + 52/377$ (Added red boxes)

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Exercise Solution IV

- $C_1 = 52/377 - y_0 - v_0$ → C_1
- Find C_2 from first red box
- $C_2 = y_0 - C_1 + 23/377 = y_0 - [52/377 - y_0 - v_0] + 23/377 = 2y_0 + v_0 - 29/377$
- $y = y_H + y_P = C_1e^{-2x} + C_2e^{-x} - 23\cos(5x)/377 + 15\sin(5x)/377$
- $y = [52/377 - y_0 - v_0]e^{-2x} + [2y_0 + v_0 - 29/377]e^{-x} - 23\cos(5x)/377 + 15\sin(5x)/377$

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Check Initial Conditions I

- $y = [52/377 - y_0 - v_0]e^{-2x} + [2y_0 + v_0 - 29/377]e^{-x} - 23\cos(5x)/377 + 15\sin(5x)/377$
- $y(0) = [52/377 - y_0 - v_0]e^{-0} + [2y_0 + v_0 - 29/377]e^{-0} + 23\cos(0)/377 + 15\sin(0)/377 = (52 - 29 - 23)/377 - y_0 - v_0 + 2y_0 + v_0 = y_0$ Correct y(0)!
- $y' = -2[52/377 - y_0 - v_0]e^{-2x} - [2y_0 + v_0 - 29/377]e^{-x} + 115\cos(5x)/377 + 75\sin(5x)/377$

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Check Initial Conditions II

- $y' = -2[52/377 - y_0 - v_0]e^{-2x} - [2y_0 + v_0 - 29/377]e^{-x} + 115\sin(5x)/377 + 75\cos(5x)/377$
- $y'(0) = -2[52/377 - y_0 - v_0]e^{-0} - [2y_0 + v_0 - 29/377]e^{-0} + 115\sin(0)/377 + 75\cos(0)/377$
- $y'(0) = -2[52/377 - y_0 - v_0] - [2y_0 + v_0 - 29/377] + 75/377 = [-104 + 29 + 75]/377 + 2y_0 + 2v_0 - 2y_0 - v_0 = v_0$ Correct y'(0)!

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